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Weibull statistics for expanded ring flexure tests of ceramics

S. G. SESHADRI, M. SRINIVASAN
The Caborundum Company, Niagara Falls, New York 14302, USA

An analysis has been developed for obtaining tensile strength distribution parameters from expanded ring flexure tests by considering the non-uniform stress distribution which occurs in the test. It is shown that expanded ring flexure strengths overestimate the tensile strength by approximately 5 to 10%.

1. Introduction

Ceramic materials are generally brittle and nominally identical batches of specimens may exhibit large variations in fracture stress. In order to compare strength values obtained from laboratory tests using a limited number of specimens of identical size and shape for actual design purposes, it is necessary to establish adequate statistical characterization of the strength distributions. It is generally known that fracture of brittle materials is caused by tensile stresses applied to the intrinsic crack-like flaws in the material. On the other hand, compressive forces resist crack opening. Thus the characterization of tensile strength distribution becomes essential for design with ceramics including lifetime predictions for components in service.

Weibull statistics developed for isotropic, homogeneous materials are widely accepted as representing strength distributions of ceramics. This analysis contends that the strength distributions can be fully described by three material parameters: a shape parameter, m (Weibull modulus); a scale parameter, σ_0 (characteristic strength); and a minimum strength parameter, σ_u. The failure probability of a body in a particular stressed state is obtained from these parameters by first summing the risks of rupture of every element in the body and calculating the total failure probability of the material. These parameters are generally evaluated from various laboratory testing techniques. Tensile testing of ceramics is experimentally difficult to perform because their low toughness results in grip failures so that other easier testing techniques like flexure tests and expanded ring tests are used to evaluate Weibull parameters. The non-uniform stress distributions existing in the specimens during the former tests lead to strength values (maximum applied stresses) greater than the actual tensile strengths. The expanded ring tests are closer to the uniaxial tensile test in estimating brittle material strengths for reasons discussed later and appear to be more realistic in predicting failure at lower stress levels. In this paper, an analytical derivation of Weibull statistics applied to expanded ring tests has been developed for both volume controlled and surface controlled flaw populations.

2. Weibull theory

When the strength of an isotropic, homogeneous brittle material is governed by volumetric flaw population, Weibull theory predicts the probability of failure at a given stress state as

$$P = 1 - \exp - \left[\int_V \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m \mathrm{d}V \right] \quad \text{for } \sigma > \sigma_u$$

and

$$= 0 \quad \text{for } \sigma \leqslant \sigma_u, \tag{1}$$

where the integral term is known as risk-of-rupture. When the strength is controlled by the surface flaw population, the integral is performed over the surface area of the body and σ represents the two-dimensional stress distributions at the surface. It may be noted that this theory also accommodates the size effects for the material through the surface and volume terms in Equation 1. In the case of uniform tension, the integral can be evaluated easily. When a non-uniform stress distribution is present, this integral must be performed over the total specimen volume (or area). Such calculations

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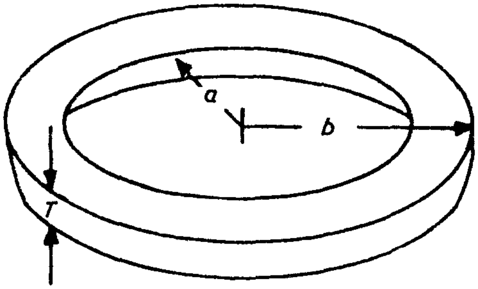

Figure 1 Specimen configuration for expanded ring flexure tests.

for bending tests have already been performed by Weil and Daniel [1].

3. Analysis for an expanded ring in tension

The specimen configuration for the expanded ring test is shown in Fig. 1. The circumferential tensile stress distribution is much greater than the radial stress distribution and is assumed to determine failure in these specimens. The stress distribution is given by

$$\sigma = \sigma_m \frac{[1 + (b/r)^2]}{[1 + (b/a)^2]}, \qquad (2)$$

where b and a are, respectively, the outer and inner radii, r is the distance from the centre and σ_m is the maximum stress in the specimen. σ_m occurs at $r = a$ and is given by

$$\sigma_m = \frac{P_i[1 + (b/a)^2]}{(b/a)^2 - 1}, \qquad (3)$$

where P_i is the applied internal pressure. If σ_u is assumed to be the minimum strength of the material, the radial distance, r_u, at which $\sigma = \sigma_u$ can be obtained from Equation 2.

$$\sigma_u = \sigma_m \frac{1 + (b/r_u)^2}{1 + (b/a)^2}. \qquad (4)$$

The radial distance at which minimum strength occurs, r_u, may be greater or less than b. When $r_u > b$, all the specimen volume (or surface) contributes to the risk-of-rupture and when $r_u < b$ only the volume (or surface) within the bounds $r = a$ and $r = r_u$ contributes to the risk-of-rupture. In the two-parameter case, σ_u becomes zero (or r_u is non-existent). This condition is reflected in the complex situation (from Equation 4)

$$\left(\frac{b}{r_u}\right)^2 = -1. \qquad (5)$$

The three-parameter Weibull analysis can be carried out for both volumetric and surface flaw conditions and two-parameter results can be easily obtained by invoking the condition that $(b/r_u)^2 = -1$.

3.1. Volumetric flaw consideration

The risk-of-rupture, R, is given by

$$R = \int_V \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m \mathrm{d}V \qquad (6)$$

and

$$V = \pi(b^2 - a^2)t \qquad (7)$$

where t is the thickness.

$$R = \int_a^{r_e} \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m 2\pi t r \mathrm{d}r$$

$$= \frac{2V}{(b^2 - a^2)} \frac{a^{2m}}{(b^2 + a^2)^m} \left(\frac{\sigma_m}{\sigma_0}\right)^m$$

$$\times \int_a^{r_e} [(b/r)^2 - (b/r_u)^2]^m \, r\mathrm{d}r, \qquad (8)$$

where $r_e = r_u$ when $r_u < b$ and $r_e = b$ when $r_u > b$. The risk-of-rupture can be expressed as

$$R = k_v V \left(\frac{\sigma_m - \sigma_u}{\sigma_0}\right)^m, \qquad (9)$$

where k_v is defined as the loading factor and

$$k_v = \frac{2}{(b^2 - a^2)[(b/a)^2 - (b/r_u)^2]^m}$$

$$\times \int_a^{r_e} [(b/r)^2 - (b/r_u)^2]^m \, r\mathrm{d}r. \qquad (10)$$

The loading factor for two-parameter distribution is obtained by invoking the conditions that $(b/r_u)^2 = -1$ and $r_e = b$. Then

$$k_v = \frac{2}{(b^2 - a^2)[(b/a)^2 + 1]^m}$$

$$\times \int_a^b [1 + (b/r)^2]^m \, r\mathrm{d}r. \qquad (11)$$

3.2. Surface flaw consideration

The risk-of-rupture integral is written as

$$R = \int_s \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m \mathrm{d}s. \qquad (12)$$

This surface integral can be divided into three parts, each corresponding to inner cylindrical surface, outer cylindrical surface and two end flat surfaces, respectively.

3.2.1. Inner cylindrical surface

The applied stress on this surface is uniform and is given by

$$\sigma = \sigma_m.$$

The risk-of-rupture for this stress state is given by

$$R_1 = 2\pi at \left(\frac{\sigma_m - \sigma_u}{\sigma_0}\right)^m. \tag{13}$$

3.2.2. Outer cylindrical surface

This surface contributes to risk-of-rupture only if $r_u > b$. For $r_u > b$,

$$\sigma = \sigma_m \frac{2}{[1 + (b/a)^2]}$$

and

$$R_2 = 2\pi bt \frac{[1 - (b/r_u)^2]^m}{[(b/a)^2 - (b/r_u)^2]^m} \left[\frac{\sigma_m - \sigma_u}{\sigma_0}\right]^m. \tag{14}$$

For $r_u < b$,

$$R_2 = 0. \tag{15}$$

Equations 14 and 15 can be combined and written as

$$R_2 = 2\pi bt \left[\frac{(b/r_e)^2 - (b/r_u)^2}{(b/a)^2 - (b/r_u)^2}\right]^m \left(\frac{\sigma_m - \sigma_u}{\sigma_0}\right)^m. \tag{16}$$

3.2.3. End flat surfaces

The two end flat surfaces are subjected to non-uniform stress distribution. The risk-of-rupture expression is similar to Equation 8.

$$R_3 = 4\pi \frac{\left[\int_a^{r_e} [(b/r)^2 - (b/r_u)^2]^m \, r\mathrm{d}r\right]}{[(b/a)^2 - (b/r_u)^2]^m} \times \left(\frac{\sigma_m - \sigma_u}{\sigma_0}\right)^m. \tag{17}$$

The total risk-of-rupture for all surfaces is given by

$$R = R_1 + R_2 + R_3$$

$$= k_s S \left(\frac{\sigma_m - \sigma_u}{\sigma_0}\right)^m, \tag{18}$$

where S is the total specimen area, and

$$S = 2\pi[at + bt + b^2 - a^2].$$

The loading factor k_s for three-parameter case can then be written as

$$k_s = \frac{1}{(at + bt + b^2 - a^2)} \times \left\{ at + bt \left[\frac{(b/r_e)^2 - (b/r_u)^2}{(b/a)^2 - (b/r_u)^2}\right]^m + \frac{2}{(b/a)^2 - (b/r_u)^2} \times \int_a^{r_e} [(b/r_u)^2 - (b/r_u)^2]^m \, r\mathrm{d}r \right\}. \tag{19}$$

Equation 19 can then be reduced to the two-parameter case by equating $(b/r_u)^2 = -1$ and $r_e = b$.

$$k_s = \frac{1}{(at + bt + b^2 - a^2)} \times \left\{ at + bt \left[\frac{2}{1 + (b/a)^2}\right]^m + \frac{2}{1 + (b/a)^2} \times \int_a^{r_e} [1 + (b/r)^2]^m \, r\mathrm{d}r \right\}. \tag{20}$$

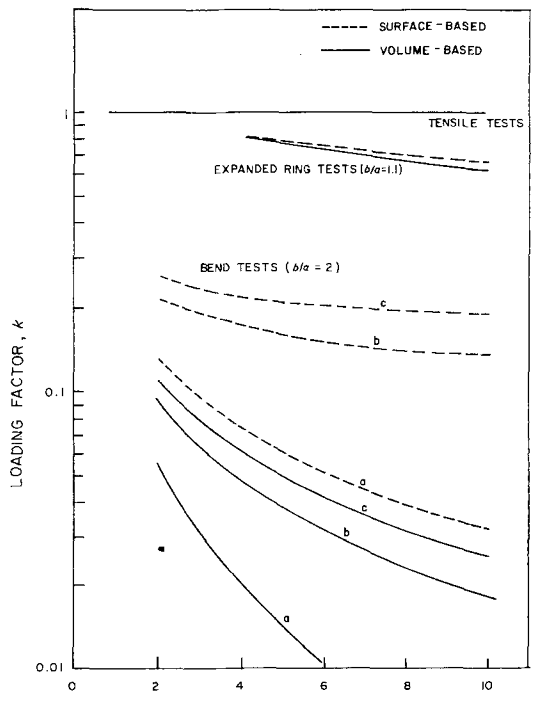

Figure 2 Loading factors for various testing methods: (a) Centre point loading; (b) third point loading (four-point bending); (c) fourth point loading (four-point bending).

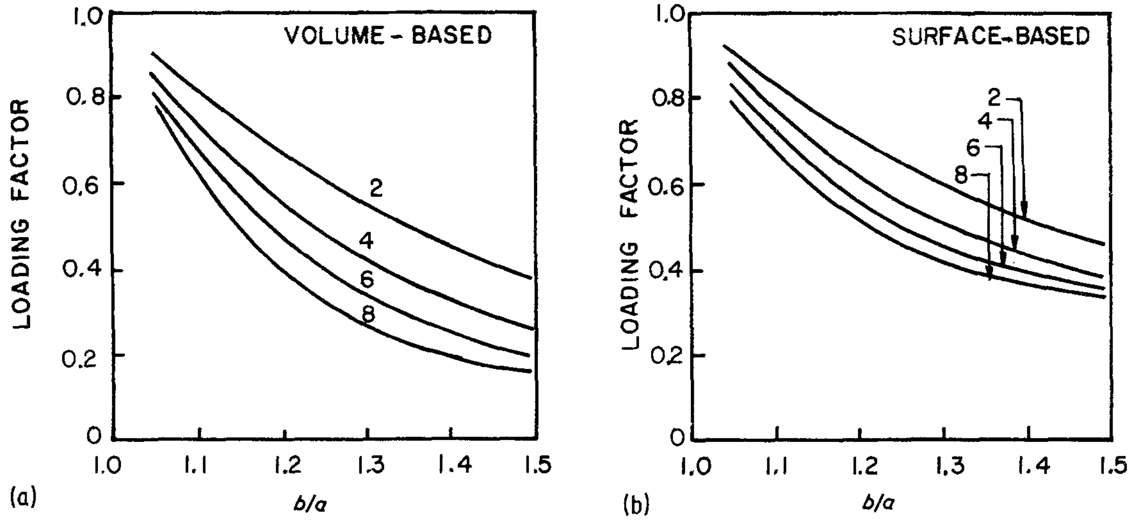

Figure 3 Loading factors for two-parameter Weibull form ($\sigma_u = 0$) for Weibull modulus, m, values indicated on the curves: (a) volume-based and (b) surface-based.

4. Discussion

The loading factors for an expanded ring test are functions of Weibull modulus, m, specimen dimensions and the ratio of minimum strength parameter to the maximum applied stress in each specimen. The fact that the loading factors are less than unity for all practical conditions differentiates this test from tensile tests where the loading factor is unity. However, the deviation of the expanded ring flexure strength from the tensile strength is less significant than for bend tests as shown later. Although this test simulates uniaxial tensile testing conditions much better than any other test methods (excluding tensile testing), the nomenclature of tensile strengths for expanded ring flexure strengths is misleading [2, 3]. The loading factors for various test conditions for practical specimen configurations are shown with respect to Weibull modulus values in Fig. 2. One can appreciate how closely the expanded ring tests simulate uniaxial tensile conditions compared to various flexure tests. The expanded ring flexure strengths will overestimate the tensile strengths of most ceramics by about 4 to 8% compared to the bend tests which overestimate the tensile strength by more than 30 to 40%. In other words, extrapolation into the low tensile strength range for obtaining the low failure probability of brittle components can introduce

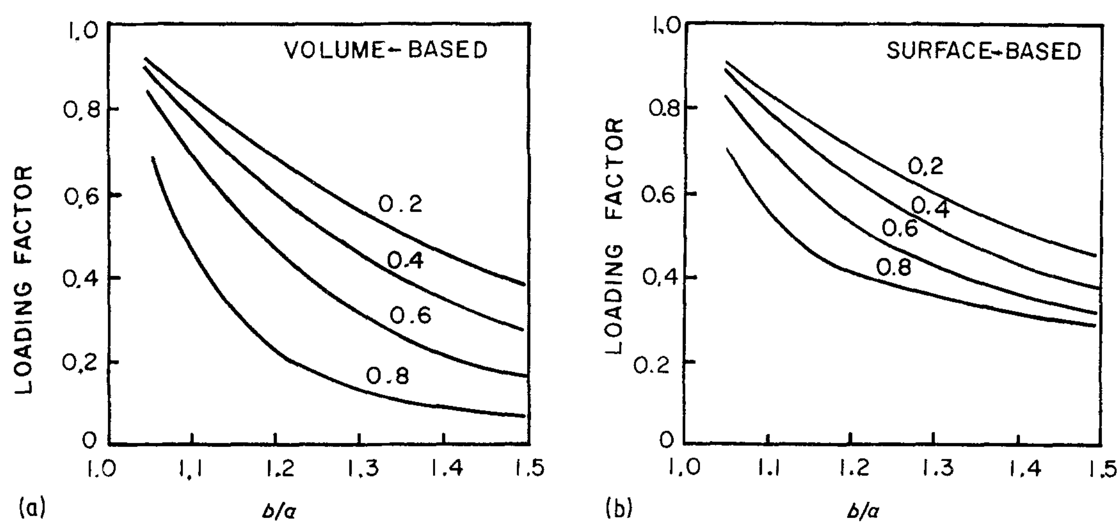

Figure 4 Loading factors for three-parameter Weibull form for a Weibull modulus, m, of 3 and α values indicated on the curves ($\alpha = \sigma_u/\sigma_{max}$): (a) volume-based and (b) surface-based.

TABLE I

Material	Flaw population	Distribution type	Estimate of Weibull parameters	
			Corrected	From Jones and Rowcliffe [3]
Hot-pressed silicon nitride Billet A	Volume	$\sigma_u = 0$	$m = 8.9$	9.4
$a = 25.4$ mm			$\sigma_0 = 425$ MPa	450 MPa
$b = 27.9$ mm			$R = 0.9889$	–
$t = 7.6$ mm		$\sigma_u = 47$ MPa	$m = 7.8$	–
			$\sigma_0 = 378$ MPa	–
			$R = 0.9905$	–
	Surface	$\sigma_u = 0$	$m = 8.9$	9.4
			$\sigma_0 = 595$ MPa	613.0 MPa
			$R = 0.9889$	
		$\sigma_u = 47$ MPa	$m = 7.8$	–
			$\sigma_0 = 553$ MPa	–
			$R = 0.9906$	
Hot-pressed silicon nitride Billet B	Volume	$\sigma_u = 0$	$m = 9.7$	11.0
$a = 25.4$ mm			$\sigma_0 = 391$ MPa	412 MPa
$b = 27.9$ mm			$R = 0.9876$	–
$t = 7.6$ mm		$\sigma_u = 125$ MPa	$m = 6.3$	–
			$\sigma_0 = 265$ MPa	–
			$R = 0.9882$	–
	Surface	$\sigma_u = 0$	$m = 9.7$	11.0
			$\sigma_0 = 533$ MPa	537 MPa
			$R = 0.9876$	–
		$\sigma_u = 108$ MPa	$m = 6.8$	–
			$\sigma_0 = 438$ MPa	–
			$R = 0.9894$	–
CVD SiC	Volume	$\sigma_u = 0$	$m = 4.9$	5.4
$a = 25.4$ mm			$\sigma_0 = 310$ MPa	312 MPa
$b = 28.45$ mm			$R = 0.9967$	–
$t = 12.7$ mm		$\sigma_u = 20$ MPa	$m = 4.5$	–
			$\sigma_0 = 291$ MPa	–
			$R = 0.9957$	–
	Surface	$\sigma_u = 0$	$m = 4.9$	5.4
			$\sigma_0 = 540$ MPa	514 MPa
			$R = 0.9967$	–
		$\sigma_u = 20$ MPa	$m = 4.5$	–
			$\sigma_0 = 534$ MPa	–
			$R = 0.9959$	–

significant errors if bend test results are used for constructing tensile strength distributions. The expanded ring test data will alleviate this problem to a great extent.

It is also seen in Fig. 2 that the load factor differences between the bend tests and the ring tests increase rapidly as the Weibull modulus increases. However, the ratio of the corresponding strengths is far less affected by the Weibull modulus as given by

$$\left(\frac{\sigma_1}{\sigma_2}\right) = \left(\frac{k_2}{k_1}\right)^{1/m}, \tag{21}$$

where σ_1, σ_2, k_1 and k_2 represent the strengths and loading factors for two testing techniques.

The loading factors for a two-parameter case are much simpler (Equations 11 and 20), especially for volumetric flaw-controlled strength distributions where k_v is a function of Weibull modulus, m, and the ratio of the radii of the specimen, (b/a).

TABLE II

Material	Flaw population	Estimate of Weibull distribution parameters			
		σ_m	σ_0 (MPa)	σ_u (MPa)	R
RBSN (small)	Volume	7.1	118	–	0.9616
$a = 25.4$ mm		2.4	52	66	0.9797
$b = 30.5$ mm					
$t = 7.6$ mm	Surface	7.1	167	–	0.9616
		2.2	140	71	0.9825
RBSN (large)	Volume	7.0	120	–	0.9150
		1.1	83	71	0.9598
	Surface*	7.0	160	–	0.9150

*The data does not converge to give a meaningful estimate of the distribution parameters with σ_u other than zero.

The surface flaw conditions also include the absolute value of one of the radii and the thickness of the specimen as additional variables that affect the load factor k_s. The variation of the load factor with respect to ratio of the radii and Weibull modulus for fixed conditions such that inner radius and thickness of the specimen are equal to unity, is shown as a family of curves in Fig. 3. Again, the deviation of the load factor from unity is clearly seen. For a given condition, the surface load factor k_s is greater than the volumetric load factor k_v.

In the three-parameter case, one more variable needs to be considered: the ratio of minimum strength to maximum applied stress (flexural strength) of each specimen. Fig. 4 shows the relationship between these variables for particular values of Weibull modulus and specimen dimensions.

5. Application to test results

The Weibull parameters can be estimated from a test population by an iterative regression procedure. For surface and volume flaw considerations, the Weibull distributions can be written, respectively, as

$$\ln \ln \left[\frac{1}{(1-P)}\right] = m \ln (\sigma_m - \sigma_u) + \ln k_v + \ln \left(\frac{V}{\sigma_0^m}\right) \tag{22}$$

and

$$\ln \ln \left[\frac{1}{(1-P)}\right] = m \ln (\sigma_m - \sigma_u) + \ln k_s + \ln \left(\frac{S}{\sigma_0^m}\right). \tag{23}$$

For the two-parameter case, $\sigma_u = 0$. A least-square fit between $\ln \ln (1/1-P)$ and $\ln \sigma_m$ provides a slope equal to m. From the intercept one can calculate σ_0 by using appropriate values of m and k. The σ_u is then assumed and a linear regression is carried out between $\ln \ln (1/1-P) - \ln k$ and $\ln (\sigma_m - \sigma_u)$. Initially $\ln k$ is assumed to be zero to obtain an estimate of m. This step is carried out iteratively by updating the load factor for each m estimate until an insignificant change in these parameters is obtained. This whole procedure is repeated to optimize the minimum strength estimate (taken to correspond to the best curve fit, as indicated by the maximum in the correlation coefficient for the regression by logically varying the assumed minimum strength value.)

Jones and Rowcliffe [3] have obtained extensive data on silicon-based ceramics by the expanded ring flexure testing technique. They estimated the Weibull parameters by assuming uniform stress distribution in the specimen with the strength equalling the maximum fracture stress in the specimen at the inner surface. This translates to the assumption of a loading factor of unity. Our corrected estimates of Weibull parameters are compared with their results in Table I. It should be noted that σ_0 has the same units as stress and the loading factors have either reciprocal volume or surface area units. Jones and Rowcliffe [3] indicated that some of the results on reaction-bonded silicon nitride (RBSN) were better described by three-parameter distributions than with $\sigma_u = 0$. They did not give any estimates of Weibull parameters for RBSN. We calculated both two- and three-parameter estimates for their data on RBSN. The results are given in Table II.

In general, the three-parameter analysis improves the regression coefficient (except in the case of chemical vapour deposited (CVD) SiC). Of course, this improvement alone does not justify the use of the three-parameter distribution. Another observation is that surface flow considerations seem to fit the data better analytically and this finding is supported by the fractographic observations made by Jones and Rowcliffe [3] that fracture mostly originated at the surface of the specimen.

6. Conclusions

The expanded ring flexure tests offer a good alternative to tensile tests, but the statistical distribution parameters must be corrected for non-uniform stress distribution which exists in the test specimen. An analysis and algorithm for obtaining these parameters is provided and has been applied to published literature data for silicon-based ceramics.

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